

## NOTE

# A Numerical Algorithm of the Multiple Scattering from an Ensemble of Arbitrary Scatterers

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### I. INTRODUCTION

Sound scattering by a finite number of arbitrary scatterers has remained a complicated problem. For this reason, only limited numerical results have been documented in the literature. The main difficulty perhaps has been associated with the lack of efficient numerical algorithms and the limited computing capability. Among many useful formalisms suggested for describing sound scattering by a finite group of arbitrary scatterers, three approaches appear to be particularly useful, that is, the self-consistent approach [1, 2], the  $T$ -matrix method [3, 4], and the method of moments [5].

The recent expanding capability of digital computers has in principle made it possible to compute the more complicated problem of multiple scattering from a finite number of scatterers. Following the self-consistent scheme in Foldy [1], the multiple scattering processes can be represented by a set of coupled linear equations. The solution to these equations can be obtained by a matrix inversion. Such a procedure has been used previously to investigate multiple scattering by isotropic scatterers, especially the acoustic localization in bubbly liquids in which air-filled bubbles are isotropic scatterers [6]. The purpose of this paper is to generalize the numerical matrix method in Ye and Alvarez [6] to more complicated cases involving many anisotropic scatterers. It will be clear from the derivation that the present approach can be used for scattering from many scatterers with arbitrary configurations for a wide range of situations.

## II. THE SELF-CONSISTENT FORMALISM

We follow the approach of Foldy [1] and Twersky [2]. Consider a wave transmitted from a projector to a number of scatterers. The scattered wave is received by a receiver, which may not be located at the same position of the projector, i.e., we consider the bistatic scattering. When the receiver is located at the position of the transmitter, it is called the backscattering. We investigate the signal at the receiver. The wave is described by the incident wavevector  $|\vec{k}_{in}| = k = \omega/c$ . Suppose that there are  $N$  scatterers in the acoustic path. The position of each scatterer is denoted as  $\vec{r}_i$  ( $i = 1, 2, \dots, N$ ).

When a wave encounters a target, it will be scattered. The scattered wave will be scattered consequently by other scatterers. This process is repeated to establish an infinitely recursive pattern of rescattering between all scatterers, causing the scattering characteristics of each scatterer to change. The multiple scattering which ensues can be conveniently treated in a self-consistent manner. The total wave reaching a receiver can be written as

$$p(\vec{r}) = p_{in}(\vec{r}) + P_s(\vec{r}), \quad (1)$$

where  $p_{in}(\vec{r})$  is the direct wave arriving at the receiver; the second term represents the total scattered wave which is a summation of all the scattered waves from each scatterer,

$$P_s(\vec{r}) = \sum_{i=1}^N p_s(\vec{r} - \vec{r}_i), \quad (2)$$

where  $p_s(\vec{r} - \vec{r}_i)$  refers to the scattered wave from the  $i$ th target.

For  $k|\vec{r} - \vec{r}_i| \gg 1$ , we may approximate the scattered wave from the  $i$ th target as

$$p_s(\vec{r} - \vec{r}_i) = p_{in}(\vec{r}_i) F_i(\theta_{\vec{r}-\vec{r}_i}, \theta_{in}) G(|\vec{r} - \vec{r}_i|) \quad (i = 1, 2, \dots, N), \quad (3)$$

where  $F_i$  is the effective scattering function of the  $i$ th target only dependent on the incidence and scattering directions and  $G(|\vec{r} - \vec{r}_i|)$  represents the usual Green's function  $\exp(i|\vec{r} - \vec{r}_i|)/|\vec{r} - \vec{r}_i|$ . Equation (3) defines the well known local far field approximation (LFFA) of the scattered field. Clearly the LFFA approximation fails when two scatterers are too close; therefore the approximation is expected to be valid for reasonably dilute many-body systems. In the later numerical example we will employ this approximation and the validity of the results will be inspected.

Without multiple scattering,  $F_i$  will be equal to the far-field single scattering function of the single target  $f_i(\theta_{\vec{r}-\vec{r}_i}, \theta_{in})$  obtained when other targets are absent. The far-field single scattering function  $f_i(\theta_{\vec{r}-\vec{r}_i}, \theta_{in})$  is relatively easy to compute for scatterers with different size and shape by several theoretical and numerical methods [3, 7, 8]. Note here that  $F_i(\theta_{\vec{r}-\vec{r}_i}, \theta_{in})$  is the effective far-field scattering function of the  $i$ th target in the direction  $\theta_{\vec{r}-\vec{r}_i}$  when the incidence is along the direction of  $\vec{k}_{in}$  including all multiple scattering from other targets.

The scattered wave from the  $i$ th target is a linear response to the total incident wave pinging on the target, which includes the direct incident wave and the scattered wave from

other targets. By Eq. (3), the scattered wave can be therefore written as

$$p_s(\vec{r} - \vec{r}_i) = \left[ f_i(\theta_{\vec{r}-\vec{r}_i}; \theta_{in}) p_{in}(\vec{r}_i) + \sum_{j=1, j \neq i}^N p_{in}(\vec{r}_j) f_i(\theta_{\vec{r}-\vec{r}_i}; \theta_{\vec{r}_i-\vec{r}_j}) \right. \\ \left. \times F_j(\theta_{\vec{r}_i-\vec{r}_j}; \theta_{in}) G(|\vec{r}_i - \vec{r}_j|) \right] G(|\vec{r} - \vec{r}_i|). \quad (4)$$

Equating Eqs. (3) and (4), we get

$$F_i(\theta_{\vec{r}-\vec{r}_i}; \theta_{in}) = f_i(\theta_{\vec{r}-\vec{r}_i}; \theta_{in}) \\ + \sum_{j=1, j \neq i}^N \frac{p_{in}(\vec{r}_j)}{p_{in}(\vec{r}_i)} f_i(\theta_{\vec{r}-\vec{r}_i}; \theta_{\vec{r}_i-\vec{r}_j}) F_j(\theta_{\vec{r}_i-\vec{r}_j}; \theta_{in}) G(|\vec{r}_i - \vec{r}_j|). \quad (5)$$

The second term on the right hand side refers to the multiple scattering effects. Notice that Eq. (5) expresses the complicated multiple scattering in terms of the scattering function of each individual scatterers that, in principle, can be easily computed for scatterers of different shape.

Setting  $\vec{r}$  at the targets except the  $i$ th, we obtain  $N - 1$  equations. In each of these  $N - 1$  equations, we allow  $i$  to vary from 1 to  $N$ . Then we have another  $N$  equations for each  $l$ . In total we have  $(N - 1) \times N$  equations for  $N(N - 1)$  unknown coefficients  $F_i(\theta_{\vec{r}_l-\vec{r}_i}; \theta_{in})$ .

Once the effective functions  $F_i(\theta_{\vec{r}_j-\vec{r}_i}; \theta_{in})$  are obtained, the scattered waves can be obtained from Eq. (4). The total wave can be subsequently obtained from Eq. (1).

### III. NUMERICAL ALGORITHM

To simplify the notation, we write Eq. (5) as

$$F_{i,l} = f_{i,l} + \sum_{j=1}^{N'} a_{i,l,j,i} F_{j,i}, \quad (6)$$

where the summation is made for  $j = 1, 2, \dots, N$  excluding  $j = i$ , and we denote  $F_{i,l} = F_i(\theta_{\vec{r}_l-\vec{r}_i}; \theta_{in})$ ,  $f_{i,l} = f_i(\theta_{\vec{r}_l-\vec{r}_i}; \theta_{in})$ , and

$$a_{i,l,j,i} = \frac{p_{in}(\vec{r}_j)}{p_{in}(\vec{r}_i)} f_i(\theta_{\vec{r}_l-\vec{r}_i}; \theta_{\vec{r}_i-\vec{r}_j}) G(|\vec{r}_i - \vec{r}_j|).$$

The equations in (6) can be rewritten in a matrix form as

$$\mathbf{F} = \mathbf{f} + \mathbf{Z} \cdot \mathbf{F} \quad (7)$$

with  $\mathbf{F} = (F_{1,2}, \dots, F_{1,N}, \dots, F_{N,1}, \dots, F_{N,N-1})$ ,  $\mathbf{f} = (f_{1,2}, \dots, f_{1,N}, \dots, f_{N,1}, \dots, f_{N,N-1})$ , and the  $N(N - 1) \times N(N - 1)$  scattering matrix  $\mathbf{Z}$ . The solution for  $\mathbf{F}$  is obtained as

$$\mathbf{F} = (1 - \mathbf{Z})^{-1} \mathbf{f}. \quad (8)$$

Therefore the solution for  $\mathbf{F}$  is obtained by an inversion of matrix  $\mathbf{Z}$ .

Due to the complicated structure, it is not trivial to build matrix  $\mathbf{Z}$ . Particularly when the number of scatterers is large, building matrix  $\mathbf{Z}$  is an arduous task because of its high dimensionality and no apparent order of its elements. Because of this, the previous studies based on this method were necessarily limited to simple situations such as isotropic scatterers or a small number of anisotropic scatterers. We propose a numerical algorithm which allows us to build the  $\mathbf{Z}$  matrix in a simple way. The core of the method is to find a mapping between elements  $F_{l,i}$  and the corresponding coefficients  $a_{i,j,l,i}$  with  $l = 1, 2, \dots, N; l \neq i$ .

A Fortran program for constructing the vectors  $\mathbf{F}$  and  $\mathbf{f}$  and the matrix  $\mathbf{Z}$  is given as

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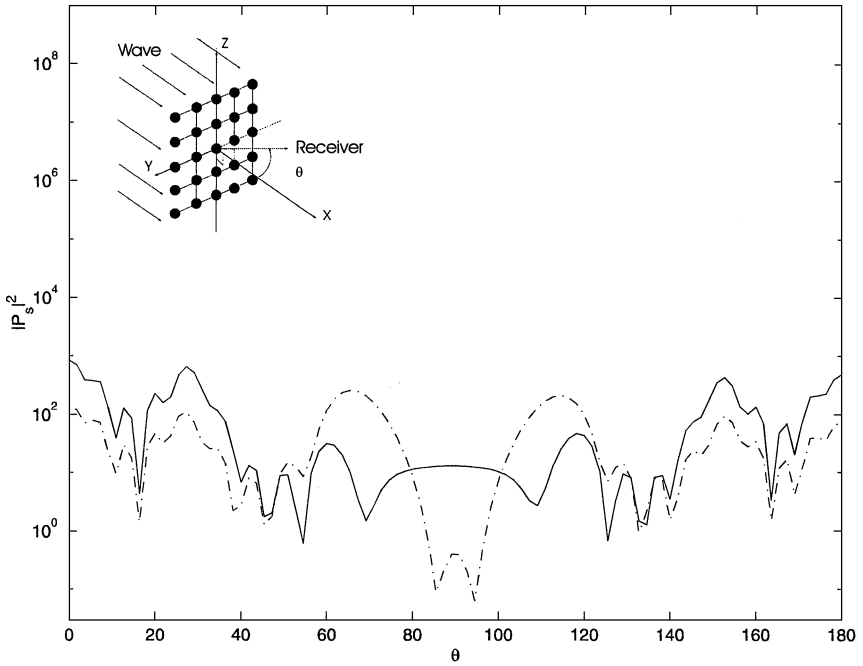
do i = 1, N
  do l = 1, N
    if (i ≠ l) then
      if (i < l) then
        m = (l - i) + (i - 1)N
      else
        m = (l - i) + (i - 1)N + 1
      end if
       $\hat{F}_m = F_{i,l} \leftarrow$  the  $\mathbf{F}$  vector
       $\hat{f}_m = f_{i,l,\theta_{in}} \leftarrow$  the  $\mathbf{f}$  vector
      do j = 1, N
        if (j ≠ i) then
          if (i > j) then
            n = (i - j) + (j - 1)N
          else
            n = (i - j) + (j - 1)N + 1
          end if
           $Z(m, n) = a_{i,l,j,i} \leftarrow$  the  $\mathbf{Z}$  matrix
        end if
      end do
    end do
  end do
end do

```

with  $m$  and  $n$  being the position indexes of the different coefficients  $a_{i,l,j,i}$  in the scattering matrix  $\mathbf{Z}$ . Once the  $\mathbf{F}$  vector is computed from Eq. (8), the final scattered wave can be obtained from Eqs. (4), (3), and (1).

#### IV. NUMERICAL APPLICATION

We now apply the numerical algorithm developed in the last section to compute the scattering from an ensemble of scatterers. For simplicity, we consider a special situation described by 25 rigid spheres of radius  $a$  which are arranged in a plane, forming a 5 by 5 square lattice. The separation between the spheres is  $d$ , which is set to  $7a$ . A unit plane wave is incident perpendicularly on the lattice structure. The frequency of the incident wave is chosen such that  $ka = 2$ ; thus  $kd = 14$ . We have numerically tested that for this separation between the rigid spheres, the LFFA approximation holds. The geometric layout is included in Fig. 1.



**FIG. 1.** Geometric layout (upper corner) and the square modulus of the scattered wave of a group of rigid spheres, as a function of scattering angles. The solid and dashed-dotted lines refer to with and without multiple scattering.

We compute the modulus of the total scattered wave, i.e.,  $|P_s(\vec{r})|^2$ , from Eq. (2). The observation is made at  $|\vec{r}| = 100a$ . The angular dependence of the total scattered field has been computed. For simplicity, we consider scattering in plane with the incident wave. The scattering angle varies from 0 to 180 degrees. Figure 1 shows the results. In the figure, the solid line refers to the results including multiple scattering and the dashed line to the results without multiple scattering. It is clear that the effects of multiple scattering are not negligible. These effects appear at all scattering angles and maximize around  $\theta = 60$  and 120 degrees.

In summary, this note presents a numerical algorithm to compute the acoustic field scattered from an ensemble of scatterers incorporating all multiple scattering. An advantage of this algorithm lies in its simplicity and generality that it may reduce much tedious computation by expressing the complicated multiple scattering in terms of the scattering function of each individual scatterer.

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